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LETTER TO THE EDITOR

An approach for solving the two-dimensional sine-Gordon equation

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Abstract. A new approach for finding analytical solutions of the two-dimensional sine-Gordon equation is presented. The essence of this approach is the established relation between the solutions of the one-dimensional wave equation having the form of running waves and solutions of the two-dimensional sine-Gordon equation.

The one-dimensional sine-Gordon equation

$$\frac{\partial^2\psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\sin\psi \qquad \psi=\psi(x,t) \tag{1}$$

has arisen in differential geometry [1]. Now the importance of the sine-Gordon equation is due to its physical applications [2] and to the fact that it has soliton solutions [1, 3, 4].

The methods already known for solving equation (1) cannot be used for finding a solution of the two-dimensional sine-Gordon equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\sin\psi \qquad \psi=\psi(x,y,t). \tag{2}$$

Here we present an approach for finding a solution of equation (2).

The substitution

$$\psi=4\tan^{-1}\sigma \qquad \sigma=\sigma(x,y,t) \tag{3}$$

leads to the following nonlinear partial differential equation for σ:

$$(1+\sigma^2)\left[\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2\sigma}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2\sigma}{\partial t^2}\right]-2\sigma\left[\left(\frac{\partial\sigma}{\partial x}\right)^2+\left(\frac{\partial\sigma}{\partial y}\right)^2-\frac{1}{c^2}\left(\frac{\partial\sigma}{\partial t}\right)^2\right]=\sigma-\sigma^3. \tag{4}$$

One possibility for splitting equation (4) into two is the following:

$$\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2\sigma}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2\sigma}{\partial t^2}=\sigma \tag{5}$$

$$\left(\frac{\partial\sigma}{\partial x}\right)^2+\left(\frac{\partial\sigma}{\partial y}\right)^2-\frac{1}{c^2}\left(\frac{\partial\sigma}{\partial t}\right)^2=\sigma^2. \tag{6}$$

Finding a solution of the system (5), (6) provides a sufficient condition for the existence of a solution of equation (2).

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Equation (6) can be split into two equations as follows:

$$\left(\frac{\partial\sigma}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\sigma}{\partial t}\right)^2 = 0 \qquad (7)$$

$$\left(\frac{\partial\sigma}{\partial y}\right)^2 = \sigma^2. \qquad (8)$$

Hence

$$\frac{\partial\sigma}{\partial x} = s\frac{1}{c}\frac{\partial\sigma}{\partial t} \qquad s = \pm 1 \qquad (9)$$

$$\frac{\partial\sigma}{\partial y} = s_1\sigma \qquad s_1 = \pm 1. \qquad (10)$$

The system (9), (10) consists of linear partial differential equations.
Equation (10) gives

$$\sigma = f(x, t)\, e^{s_1 y}. \qquad (11)$$

Substituting (11) into (9) we obtain

$$\frac{\partial f}{\partial x} = s\frac{1}{c}\frac{\partial f}{\partial t}. \qquad (12)$$

Hence

$$f(x, t) = f(x - sct). \qquad (13)$$

Inserting (13) and (11) into (5), we see that (5) is also satisfied.
So for any function $f(x - sct)$, $s = \pm 1$, the function

$$\psi(x, y, t) = 4\tan^{-1}[f(x - sct)\, e^{s_1 y}] \qquad (14)$$

is a solution of equation (2).

The procedure presented here gives a simple relation between the solutions of the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

having the form of running waves $f(x + ct)$ or $f(x - ct)$ and solutions of the two-dimensional sine-Gordon equation (2) having the form (14).

We see from (14) that the amplitude of the solution of (2) is a function of the velocity c, which is a typical soliton property.

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